MODULE CATEGORIES OVER AFFINE GROUP SCHEMES

SHLOMO GELAKI

Abstract. Let k be an algebraically closed field of characteristic $p \geq 0$. Let G be an affine group scheme over k. We classify the indecomposable exact module categories over the rigid tensor category $\mathrm{Coh}_f(G)$ of coherent sheaves of finite dimensional $k-$vector spaces on G, in terms of $(H,\psi)-$equivariant coherent sheaves on G. We deduce from it the classification of indecomposable exact *geometrical* module categories over $\mathrm{Rep}(G)$. When G is finite, this yields the classification of *all* indecomposable exact module categories over the finite tensor category $\mathrm{Rep}(G)$. In particular, we obtain a classification of twists for the group algebra $k[G]$ of a finite group scheme G. Applying this to $u(\mathfrak{g})$, where $\mathfrak{g}$ is a finite dimensional $p-$Lie algebra over k with positive characteristic, produces (new) finite dimensional noncommutative and noncocommutative triangular Hopf algebras in positive characteristic. We also introduce and study group scheme theoretical categories, and study isocategorical finite group schemes by following [EG1].

1. introduction

Let k be an algebraically closed field of characteristic $p \geq 0$. Let G be a finite group. Consider the fusion category $\mathrm{Vec}(G)$ of finite dimensional $G-$graded vector spaces over k, and the finite tensor category $\mathrm{Rep}(G)$ of finite dimensional representations of G over k. Etingof and Ostrik classified the indecomposable exact module categories over $\mathrm{Rep}(G)$ [EO], generalizing the classification of Ostrik in zero characteristic [Os]. Alternatively, one could use the duality between $\mathrm{Vec}(G)$ and $\mathrm{Rep}(G)$ (provided by the usual fiber functor on $\mathrm{Rep}(G)$) and the classification of the indecomposable exact module categories over $\mathrm{Vec}(G)$ to obtain the same result. In particular, the classification of the semisimple module categories of rank 1 provides the classification of twists for the group algebra $k[G]$, reproducing the classification given by Movshev in zero characteristic [Mov]. The classification of twists for finite

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groups, together with Deligne's theorem [De], enabled Etingof and the author to classify triangular semisimple and cosemisimple Hopf algebras over k [EG] (see also [Ge]).

The goal of this paper is to extend the classification of Etingof and Ostrik mentioned above to *finite group schemes* G over k, and in particular thus obtain (new) finite dimensional noncommutative and noncocommutative triangular Hopf algebras in positive characteristic by twisting $k[G]$. However, in absence of Deligne's theorem in positive characteristic, the classification of finite dimensional triangular Hopf algebras in positive characteristic remains out of reach.

Let G be a finite group scheme over k. The idea is to first classify the indecomposable exact module categories over $\mathrm{Rep}(k[G]^*)$, where $k[G]^*$ is the dual Hopf algebra of the group algebra $k[G]$ of G, and then use the fact that they are in bijection with the indecomposable exact module categories over $\mathrm{Rep}(G)$ [EO] to get the classification of the latter ones. The reason we aproche it in this way is that $k[G]^*$ is just the Hopf algebra $\mathcal{O}(G)$ representing the group scheme G, so $\mathrm{Rep}(k[G]^*)$ is tensor equivalent to the tensor category $\mathrm{Coh}_f(G) = \mathrm{Coh}(G)$ of coherent sheaves of *finite dimensional* $k-$vector spaces on G with the tensor product of convolution of sheaves, which allows us to use geometrical tools and arguments. For example, when G is an abstract finite group, $\mathrm{Coh}(G) = \mathrm{Vec}(G)$.

In fact, in Theorem 3.7 we classify the indecomposable exact module categories over $\mathrm{Coh}_f(G)$, where G is *any* affine group scheme over k (i.e., G is not necessarily finite). The classification is given in terms of $(H, \psi)-$equivariant coherent sheaves on G (see Definition 3.2). Since $\mathrm{Coh}_f(G)$ is no longer finite when G is not, the proof requires working with Ind and Pro objects, which makes it technically more involved. Furthermore, when G is not finite, not all indecomposable exact module categories over $\mathrm{Rep}(G)$ are obtained from those over $\mathrm{Coh}_f(G)$ (see Proposition 4.2 and Remark 4.3); we refer to those which are as *geometrical*. So the classification of module categories (even fiber functors) over $\mathrm{Rep}(G)$ for infinite affine group schemes G remains unknown (even when G is a linear algebraic group over $\mathbb{C}$).

In Section 5 we introduce the class of *group scheme theoretical categories*, which extends both $\mathrm{Coh}_f(G)$ and $\mathrm{Rep}(G)$, and generalize to them the results from Sections 3 and 4 mentioned above.

As a consequence of our results, combined with [AEGN, EO], we obtain in Corollary 6.3 that gauge equivalence classes of twists for the group algebra $k[G]$ of a *finite* group scheme G over k are parameterized by conjugacy classes of pairs (H, J), where H is a closed group subscheme of G and J is a *nondegenerate* twist for $k[H]$ (just as in

the case of abstract finite groups). Furthermore, in Proposition 6.7 we show that a twist for G is nondegenerate if and only if it is *minimal* (again, as for abstract finite groups), by showing directly, that is, without using Deligne's theorem, that a quotient of a Tannakian category is also Tannakian (Proposition 6.5). We use this in Sections 6.4, 6.5 to give some examples of twists for $k[A]$ and $u(\mathfrak{g})$, where A is a finite commutative group scheme over k with positive characteristic and $\mathfrak{g}$ is a finite dimensional $p-$Lie algebra over k of positive characteristic p. In particular, applying this to $u(\mathfrak{g})$ yields (new) finite dimensional noncommutative and noncocommutative triangular Hopf algebras in positive characteristic.

We conclude with Section 7 in which we follow [EG1] to give the construction of all finite group schemes which are isocategorical to a fixed finite group scheme. In particular, it follows that two isocategorical finite group schemes are necessarily isomorphic as schemes (but not as groups [EG1], [Da1, Da2]).

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2. Preliminaries

Throughout the paper we fix an algebraically closed field k of characteristic $p \geq 0$.

2.1. **Affine group schemes.** Let G be an affine group scheme over k with coordinate algebra $\mathcal{O}(G)$, i.e., $\mathcal{O}(G)$ is a commutative Hopf algebra and $G = \mathrm{Spec}_k(\mathcal{O}(G))$ (see, e.g., [Jan]). Let G^0 be the connected component of the identity in G, and let $\pi_0(G) := G/G^0$. Then $\mathcal{O}(\pi_0(G))$ is the unique maximal finite dimensional semisimple Hopf subalgebra of $\mathcal{O}(G)$. We also let $\mathrm{Rep}(G)$ denote the category of finite dimensional rational representations of G over k; it is a *symmetric rigid tensor category* (see, e.g., [E] for the definition of a tensor category and its general theory).

An affine group scheme G is called *finite* if $\mathcal{O}(G)$ is finite dimensional. In this case, $\mathcal{O}(G)^*$ is a finite dimensional cocommutative Hopf algebra, which is called the *group algebra* of G, and denoted by $k[G]$. In particular, $\mathrm{Rep}(G)$ is a *finite* symmetric tensor category and $\mathrm{Rep}(G) = \mathrm{Rep}(k[G])$ as symmetric tensor categories. A finite group scheme G is called *constant* if its representing Hopf algebra $\mathcal{O}(G)$ is the Hopf algebra of functions on some finite abstract group with values in k, and is called *etale* if $\mathcal{O}(G)$ is semisimple. Since k is algebraically

closed, it is known that G is etale if and only if it is a constant group scheme [W]. A finite group scheme G is called *infinitesimal* if $\mathcal{O}(G)$ is a local algebra.

Theorem 2.1. *(See* [W, 6.8, p.52]*) Let G be a finite group scheme. Then $\pi_0(G)$ is etale, G^0 is infinitesimal, and G is a semidirect product $G = G^0 \rtimes \pi_0(G)$. If the characteristic of k is 0 then $G = \pi_0(G)$ is etale.*

Let G be a finite *commutative* group scheme over k, i.e., $\mathcal{O}(G)$ is a finite dimensional commutative and cocommutative Hopf algebra. In this case, $k[G]$ is also a finite dimensional commutative and cocommutative Hopf algebra, so it represents a finite commutative group scheme G^D over k, which is called the *Cartier dual* of G. For example, the Cartier dual of the group scheme α_p (= the Frobenius kernel of the additive group $\mathbb{G}_a$) is α_p, while the Cartier dual of μ_p (= the Frobenius kernel of the multiplicative group $\mathbb{G}_m$) is the constant group scheme $\mathbb{Z}/p\mathbb{Z}$.

Theorem 2.2. *(See* [W, 6.8, p.52]*) Let G be a finite commutative group scheme over k. Then $G = G_{ee} \times G_{ec} \times G_{ce} \times G_{cc}$ decomposes canonically as a direct product of four finite commutative group schemes over k of the following types: G_{ee} is etale with etale dual (i.e., an abstract abelian group A such that $p \nmid |A|$), G_{ec} is etale with connected dual (e.g., $\mathbb{Z}/p\mathbb{Z}$), G_{ce} is connected with etale dual (e.g., μ_p), and G_{cc} is connected with connected dual (e.g., $\alpha_p \cong \alpha_p^D$).*

Recall that a finite commutative group scheme G is called *diagonalizable* if $\mathcal{O}(G)$ is the group algebra $k[A]$ of a finite abelian group A. For example, μ_n is diagonalizable since $\mathcal{O}(\mu_n) = k[\mathbb{Z}/n\mathbb{Z}]$. In fact, any diagonalizable finite group scheme G is a direct product of various μ_n. Clearly, the group algebra of a finite diagonalizable group scheme is semisimple.

Theorem 2.3. *(Nagata, see* [A, p.223]*) Let G be a finite group scheme over k. The group algebra $k[G]$ is semisimple if and only if G^0 is diagonalizable and p does not divide the order of $G(k)$. In particular, for an infinitesimal group scheme G, if $k[G]$ is semisimple then G is diagonalizable.*

2.2. $p-$**Lie algebras.** Assume that the ground field k has characteristic $p > 0$. Let $\mathfrak{g}$ be a finite dimensional $p-$Lie algebra over k and let $u(\mathfrak{g})$ be its $p-$restricted universal enveloping algebra (see, e.g., [Jac], [SF]). Then $u(\mathfrak{g})$ is a cocommutative Hopf algebra of dimension $p^{\dim(\mathfrak{g})}$ and its dual (commutative) Hopf algebra $u(\mathfrak{g})^*$ is a local algebra satisfying $x^p = 0$ for any x in the augmentation ideal of $u(\mathfrak{g})^*$. Recall

that there is an equivalence of categories between the category of infinitesimal group schemes G over k of *height* 1 and the category of finite dimensional $p-$Lie algebras $\mathfrak{g}$ over k, given by $G \mapsto \mathfrak{g}$, where $k[G] = u(\mathfrak{g})$.

An $n-$dimensional *torus* is an $n-$dimensional abelian $p-$Lie algebra $\mathfrak{t}$ over k with a basis consisting of *toral* elements h_i (i.e, $h_i^p = h_i$). By a theorem of Hochschild (see Theorem 2.3), tori are precisely those finite dimensional $p-$Lie algebras whose representation categories are semisimple (see [SF]). In other words, $u(\mathfrak{t})$ is a semisimple commutative (and cocommutative) Hopf algebra, and $\mathrm{Rep}(\mathfrak{t}) = \mathrm{Rep}(u(\mathfrak{t}))$ is a fusion category. Moreover, it is known that $u(\mathfrak{t})$ is isomorphic to the Hopf algebra $\mathrm{Fun}((\mathbb{Z}/p\mathbb{Z})^n, k)$ of functions on the elementary abelian $p-$group of rank n, so $\mathrm{Rep}(u(\mathfrak{t})) = \mathrm{Vec}((\mathbb{Z}/p\mathbb{Z})^n)$ is the fusion category of finite dimensional $(\mathbb{Z}/p\mathbb{Z})^n-$graded vector spaces.

2.3. **Module categories over tensor categories.** Let $\mathcal{C}$ be a rigid tensor category over k, i.e., a $k-$linear locally finite abelian category with finite dimensional Hom$-$spaces, equipped with an associative tensor product, a unit object and a rigid structure (see, e.g., [E]).

Let $\mathrm{Ind}(\mathcal{C})$ and $\mathrm{Pro}(\mathcal{C})$ denote the categories of Ind$-$objects and Pro$-$objects of $\mathcal{C}$, respectively (see, e.g., [KS]). The rigid structure on $\mathcal{C}$ induces two duality functors $\mathrm{Pro}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C})$ ("continuous dual") and $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Pro}(\mathcal{C})$ ("linear dual"), which we shall both denote by $X \mapsto X^*$; they are antiequivalence inverses of each other.

It is well known that the tensor structure on $\mathcal{C}$ extends to a tensor structure on $\mathrm{Ind}(\mathcal{C})$ and $\mathrm{Pro}(\mathcal{C})$ (however, $\mathrm{Ind}(\mathcal{C})$ and $\mathrm{Pro}(\mathcal{C})$ are not rigid). It is also known that $\mathrm{Ind}(\mathcal{C})$ has enough injectives. More generally, recall that a (left) *module category* over $\mathcal{C}$ is a locally finite abelian category $\mathcal{M}$ over k equipped with a (left) action $\otimes^{\mathcal{M}}$ of $\mathcal{C}$ on $\mathcal{M}$, such that the bifunctor $\otimes^{\mathcal{M}}$ is bilinear on morphisms and biexact. Similarly, the $\mathcal{C}-$module structure on $\mathcal{M}$ extends to a module structure on $\mathrm{Ind}(\mathcal{M})$ over $\mathrm{Ind}(\mathcal{C})$.

One can define a *dual internal Hom* in a $\mathcal{C}-$module category $\mathcal{M}$ as follows: for $M_1, M_2 \in \mathcal{M}$, let $\overline{\mathrm{Hom}}(M_1, M_2) \in \mathrm{Pro}(\mathcal{C})$ be the pro-object representing the left exact functor

$$\mathcal{C} \to \mathrm{Vec},\ X \mapsto \mathrm{Hom}_{\mathcal{M}}(M_2, X \otimes^{\mathcal{M}} M_1),$$

i.e.,

$$\mathrm{Hom}_{\mathcal{M}}(M_2, X \otimes^{\mathcal{M}} M_1) \cong \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(\overline{\mathrm{Hom}}(M_1, M_2), X).$$

For any $M \in \mathcal{M}$, the pro-object $\overline{\mathrm{Hom}}(M,M)$ has a canonical structure of a coalgebra. If $\mathcal{M}$ is indecomposable and exact then the category $\mathrm{Comod}_{\mathrm{Pro}(\mathcal{C})}(\overline{\mathrm{Hom}}(M,M))$ of right comodules over $\overline{\mathrm{Hom}}(M,M)$ in $\mathrm{Pro}(\mathcal{C})$, equipped with its canonical structure of a $\mathcal{C}-$module category, is equivalent to $\mathcal{M}$. (This is a special case of Barr-Beck Theorem in category theory; see [EO, Theorem 3.17].). We note that in terms of internal Hom's [EO], the algebra $\underline{\mathrm{Hom}}(M,M)$ in $\mathrm{Ind}(\mathcal{C})$ is isomorphic to the dual algebra of the coalgebra $\overline{\mathrm{Hom}}(M,M)$ under the duality functor $^*:\mathrm{Pro}(\mathcal{C})\to \mathrm{Ind}(\mathcal{C})$.

Example 2.4. Let H be a Hopf algebra over k, let $\mathcal{C} := \mathrm{Rep}(H)$ be the rigid tensor category of finite dimensional representations of H over k, let $\mathcal{M} := \mathrm{Vec}$ be the standard module category over $\mathcal{C}$, and let $\delta := k$ be the trivial representation. Then $\underline{\mathrm{Hom}}(\delta,\delta) = H^\circ$ is the finite dual of H (i.e., the Hopf algebra of linear functionals on H vanishing on a finite codimensional ideal of H). Indeed, let $X \in \mathcal{C}$ and denote its underlying vector space by $\overline{X}$. On one hand,

$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(X, \underline{\mathrm{Hom}}(k,k)) = \mathrm{Hom}_{\mathrm{Vec}}(X\otimes k, k) = \mathrm{Hom}_{\mathrm{Vec}}(\overline{X}, k) = \overline{X}^*.$$

On the other hand, since X is finite dimensional, we have that

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(X, H^\circ) &= \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(X, H^*) = \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(X\otimes H, k)\\ &= \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\overline{X}\otimes_k H, k) = \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(H, \overline{X}^*) = \overline{X}^*.\end{aligned}$$

Therefore the claim follows from Yoneda's lemma.

Consequently, $\overline{\mathrm{Hom}}(\delta,\delta) = (H^\circ)^* = \widehat{H}$ is the profinite completion of H with respect to its algebra structure (i.e., $\widehat{H}$ is the inverse limit $\varprojlim H/I$ over all finite codimensional proper ideals I of H).

Definition 2.5. ([EO]) A module category $\mathcal{M}$ over $\mathcal{C}$ is called *exact* if any additive module functor $\mathcal{M}\to\mathcal{M}_1$ from $\mathcal{M}$ to any other $\mathcal{C}-$module category $\mathcal{M}_1$ is exact.

Proposition 2.6. *([EO]) Let $\mathcal{C}$ be a rigid tensor category over k and let $\mathcal{M}$ be a module category over $\mathcal{C}$. Then the following are equivalent:*

1) $\mathcal{M}$ is exact.

2) For any $M \in \mathcal{M}$ and any injective object $I \in \mathrm{Ind}(\mathcal{C})$, $I\otimes M$ is injective in $\mathrm{Ind}(\mathcal{M})$.

3) For any $M \in \mathcal{M}$ and any projective object $P \in \mathrm{Pro}(\mathcal{C})$, $P\otimes M$ is projective in $\mathrm{Pro}(\mathcal{M})$.

Proof. The proof that 1) implies 2) is exactly as the proof of [EO, Proposition 3.16] (after replacing "projective" by "injective" and "$\underline{\mathrm{Hom}}$" by "$\overline{\mathrm{Hom}}$"). More precisely, if $\mathcal{M}$ is exact then the $\mathcal{C}-$module functor

$\overline{Hom}(M,?) : \mathcal{M} \to \mathrm{Pro}(\mathcal{C})$ is exact. Therefore the functor

$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{M})}(?, I \otimes M) = \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\overline{Hom}(M,?), I)$$

is exact for any injective object I in $\mathrm{Ind}(\mathcal{C})$, so $I \otimes M$ is injective in $\mathrm{Ind}(\mathcal{M})$. (Here by $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\overline{Hom}(M,?), I)$ we mean the Hom−space $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\mathbf{1}, \overline{Hom}(M,?)^* \otimes I$.)

The proof that 2) implies 1) is exactly as the proof of [EO, Proposition 3.11] (after replacing "projective" by "injective").

Finally, 2) is equivalent to 3) by duality. □

Let $\mathcal{C}$ be a rigid tensor category over k and let $\mathcal{M}$ be a module category over $\mathcal{C}$. Following [EO], we say that two simple objects $M_1, M_2 \in \mathcal{M}$ are *related* if there exists an object $X \in \mathcal{C}$ such that M_1 appears as a subquotient in $X \otimes M_2$.

Proposition 2.7. *(*[EO]*) Let $\mathcal{C}$ be a rigid tensor category over k and let $\mathcal{M}$ be an exact module category over $\mathcal{C}$. Then the following hold:*

1) The above relation is an equivalence relation.

2) $\mathcal{M}$ decomposes into a direct sum $\mathcal{M} = \oplus \mathcal{M}_i$ of indecomposable exact module subcategories indexed by the equivalence classes of the above relation.

Proof. 1) The proof is essentially the proof of [EO, Lemma 3.8 and Proposition 3.9]. Namely, the proof that the relation is reflexive and transitive is exactly the same. Suppose that M_1 appears as a subquotient in $X \otimes M_2$, and let $E(\mathbf{1}) \in \mathrm{Ind}(\mathcal{C})$ be the injective hull of the unit object $\mathbf{1} \in \mathcal{C}$. By Proposition 2.6, $E(\mathbf{1}) \otimes X \otimes M_2$ is injective in $\mathrm{Ind}(\mathcal{M})$, hence

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{M})}(X^* \otimes E(\mathbf{1})^* \otimes M_1, M_2) = \mathrm{Hom}_{\mathrm{Ind}(\mathcal{M})}(M_1, E(\mathbf{1}) \otimes X \otimes M_2) \neq 0.$$

But this implies the existences of $Y \in \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{M}}(Y \otimes M_1, M_2) \neq 0$, which proves that the relation is also symmetric.

2) For an equivalence class i let $\mathcal{M}_i$ be the full subcategory of $\mathcal{M}$ consisting of objects all simple subquotients of which lie in i. Clearly, $\mathcal{M}_i$ is an indecomposable module subcategory of $\mathcal{M}$ and $\mathcal{M} = \oplus \mathcal{M}_i$. Furthermore, $\mathcal{M}_i$ is exact since so is $\mathcal{M}$. □

Definition 2.8. We say that an object $\delta \in \mathcal{M}$ generates $\mathcal{M}$ if for any $M \in \mathcal{M}$ there exists $X \in \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{M}}(X \otimes^{\mathcal{M}} \delta, M) \neq 0$ (equivalently, for any $M \in \mathcal{M}$ there exists $X \in \mathcal{C}$ such that M is a subquotient of $X \otimes^{\mathcal{M}} \delta$).

Corollary 2.9. *Let $\mathcal{M}$ be an indecomposable exact module category over a rigid tensor category $\mathcal{C}$ and let $\delta \in \mathcal{M}$ be a simple object. Then δ generates $\mathcal{M}$.* □

3. EXACT MODULE CATEGORIES OVER $\mathrm{Coh}_f(G)$

Let G be an affine group scheme over k, with a unit morphism $e : \mathrm{Spec}(k) \to G$, an inversion morphism $i : G \to G$ and a multiplication morphism $m : G \times G \to G$, satisfying the usual group axioms. In other words, we are given a collection of group structures on the sets $G(R) := \mathrm{Hom}_{Alg}(\mathcal{O}(G), R)$ of $R-$valued points of G, where R is a commutative algebra over k, which is functorial in R.

3.1. **The category $\mathbf{Coh}_f(G)$.** We shall denote by $\mathrm{Coh}_f(G)$ (resp., $\mathrm{Coh}(G)$) the abelian category of coherent sheaves of *finite dimensional* $k-$vector spaces on G, i.e., coherent sheaves supported on finite sets in G (resp., *all* coherent sheaves of $k-$vector spaces on G). Recall that $\mathrm{Coh}_f(G)$ (resp., $\mathrm{Coh}(G)$) is a rigid tensor category (resp., tensor category) with the convolution product

$$X \otimes Y := m_*(X \boxtimes Y)$$

as the tensor product (where m_* is the direct image functor of m). It is known that $\mathrm{Coh}_f(G)$ (resp., $\mathrm{Coh}(G)$) is tensor equivalent to the rigid tensor category $\mathrm{Rep}(\mathcal{O}(G))$ of *finite dimensional* $k-$representations of the Hopf algebra $\mathcal{O}(G)$ (resp., the tensor category of *finitely generated* $k-$representations of the Hopf algebra $\mathcal{O}(G)$). Recall also that $\mathrm{Ind}(\mathrm{Coh}_f(G))$ is the category of locally finite representations of $\mathcal{O}(G)$, i.e., representations in which every vector generates a finite dimensional subrepresentation, while $\mathrm{Ind}(\mathrm{Coh}(G)) = \mathrm{QCoh}(G)$ is the category of quasicoherent sheaves of $k-$vector spaces on G, i.e., the category of *all* $k-$representations of the Hopf algebra $\mathcal{O}(G)$.

Example 3.1. If G is a finite abstract group, i.e., a constant group scheme over k, then $\mathrm{Coh}_f(G) = \mathrm{Coh}(G)$ is nothing but the fusion category $\mathrm{Vec}(G)$ of finite dimensional $G-$graded vector spaces over k.

3.2. **Equivariant coherent sheaves.** Let H be a closed group subscheme of G and let $\mu : G \times H \to G$ be its free action on G by right translations (in other words, the free actions of $H(R)$ on $G(R)$ by right translations are functorial in R, R a commutative algebra over k). Set

$$\eta := \mu(id \times m_{|H}) = \mu(\mu \times id) : G \times H \times H \to G.$$

Let

$$\mathrm{p}^1_{GH} : G \times H \to G,\ \mathrm{p}^1_{GHH} : G \times H \times H \to G,\ \mathrm{p}^{12}_{GHH} : G \times H \times H \to G \times H$$

be the projections on G, G and $G \times H$, respectively. We clearly have that $p^1_{GH} \circ p^{12}_{GHH} = p^1_{GHH}$.

Let $\psi : H \times H \to \mathbb{G}_m$ be a normalized $2-$cocycle. Equivalently, $\psi \in \mathcal{O}(H) \otimes \mathcal{O}(H)$ is a *Drinfeld twist* for $\mathcal{O}(H)$, i.e., ψ is an invertible element satisfying the equations

$$(\Delta \otimes id)(\psi)(\psi \otimes 1) = (id \otimes \Delta)(\psi)(1 \otimes \psi), \ (\varepsilon \otimes id)(\psi) = (id \otimes \varepsilon)(\psi) = 1.$$

We let $\mathcal{O}(H)_\psi$ be the ("twisted") coalgebra with underlying vector space $\mathcal{O}(H)$ and comultiplication Δ_ψ given by $\Delta_\psi(f) := \Delta(f)\psi$, where Δ is the standard comultiplication of $\mathcal{O}(H)$.

Note that ψ (like any other regular nonvanishing function) defines an automorphism of any coherent sheaf on $H \times H$ by multiplication.

Definition 3.2. Let $\psi : H \times H \to \mathbb{G}_m$ be a normalized $2-$cocycle on a closed group subscheme H of G.

1) An $(H, \psi)-$*equivariant coherent sheaf* on G is a pair (S, λ_S), where $S \in \mathrm{Coh}(G)$ and λ_S is an isomorphism $\lambda_S : (\mathrm{p}^1_{GH})^*(S) \to \mu^*(S)$ of sheaves on $G \times H$ such that the diagram of morphisms of sheaves on $G \times H \times H$

$$\begin{tikzcd}
(\mathrm{p}^1_{GHH})^*(S) \arrow[r, "(\mathrm{p}^{12}_{GHH})^*(\lambda_S)"] \arrow[d, "(id \times m_{|H})^*(\lambda_S)"'] & \mu^*(S) \arrow[d, "(\mu \times id)^*(\lambda_S)"] \\
\eta^*(S) \arrow[r, "id \boxtimes \psi"'] & \eta^*(S)
\end{tikzcd}$$

is commutative.

2) Let (S, λ_S) and (T, λ_T) be two $(H, \psi)-$equivariant coherent sheaves on G. A morphism $\phi : S \to T$ in $\mathrm{Coh}(G)$ is said to be $(H, \psi)-$*equivariant* if the diagram of morphisms of sheaves on $G \times H$

$$\begin{tikzcd}
(p^1_{GH})^*(S) \arrow[r, "(p^1_{GH})^*(\phi)"] \arrow[d, "\lambda_S"'] & (p^1_{GH})^*(T) \arrow[d, "\lambda_T"] \\
\mu^*(S) \arrow[r, "\mu^*(\phi)"'] & \mu^*(T)
\end{tikzcd}$$

is commutative.

3) Let $\mathrm{Coh}_f^{(H,\psi)}(G)$ be the abelian category of $(H, \psi)-$equivariant coherent sheaves on G *with finite support in* G/H (i.e., sheaves supported on finitely many $H-$cosets), with $(H, \psi)-$equivariant morphisms.

Remark 3.3. If (H', ψ') is another pair consisting of a closed group subscheme H' of G and a normalized $2-$cocycle ψ' on it, we can similarly define the abelian category of $((H', \psi'), (H, \psi))-$biequivariant coherent sheaves on G by considering the free right action of $H' \times H$ on G given by $g(a, b) := a^{-1}gb$.

Example 3.4. We have that $\mathrm{Coh}_f^{(\{1\},1)}(G) = \mathrm{Coh}_f(G)$ is the regular module, and $\mathrm{Coh}_f^{(G,1)}(G) = \mathrm{Vec}$ (the simple object being the regular representation $\mathcal{O}(G)$) is the usual fiber functor on $\mathrm{Coh}_f(G)$.

Remark 3.5. 1) In [Mum, p.110], an $H-$equivariant sheaf (S, λ_S) on G is referred to as a sheaf S on G together with a *lift* λ_S of the $H-$action μ on G to S.

2) It is well known that the (geometric, hence also categorical) quotient scheme G/H exists. Let $\mathrm{Coh}_f(G/H)$ be the abelian category of coherent sheaves of *finite dimensional* $k-$vector spaces on G/H, and let $\pi : G \to G/H$ be the canonical $H-$invariant morphism. It is known that the inverse image functor $\pi^* : \mathrm{Coh}_f(G/H) \to \mathrm{Coh}_f(G)$ determines an equivalence of categories between $\mathrm{Coh}_f(G/H)$ and $\mathrm{Coh}_f^{(H,1)}(G)$, with π_*^H as its inverse (where π_*^H is the subsheaf of $H-$invariants of π_*).

This following lemma will be very useful in the sequel.

Lemma 3.6. *Let H be a closed group subscheme of an affine group scheme G over k, acting on itself and on G by right translations $\mu_H : H \times H \to H$ and $\mu_G : G \times H \to G$, respectively. Let $\iota = \iota_H : H \hookrightarrow G$ be the inclusion morphism and let ψ be a normalized $2-$cocycle on H. The following hold:*

1) The structure sheaf of H (i.e., the regular representation of $\mathcal{O}(H)$) has a canonical structure of an $(H, \psi)-$equivariant coherent sheaf on H, making it the unique (up to isomorphism) simple object of $Coh_f^{(H,\psi)}(H)$.

2) The sheaf $\iota_*\mathcal{O}(H) \in Coh(G)$ (i.e., the representation of $\mathcal{O}(G)$ on $\mathcal{O}(H)$ coming from the comorphism of ι) is a simple object in $Coh_f^{(H,\psi)}(G)$.

3) For any $X \in Coh_f(G)$ and $M \in Coh_f^{(H,\psi)}(G)$, we have that $m_*(X \boxtimes M) \in Coh_f^{(H,\psi)}(G)$.

Proof. 1) Consider the isomorphism $\varphi := (\mu_H, p^2_{HH}) : H \times H \xrightarrow{\cong} H \times H$, where $p^2_{HH} : H \times H \to H$ is the projection on the second coordinate. Clearly, $p^1_{HH} \circ \varphi = \mu_H$, so $(p^1_{HH} \circ \varphi)^*\mathcal{O}(H) = \mu_H^*\mathcal{O}(H)$. Now, multiplication by ψ defines an isomorphism

$$\mu_H^*\mathcal{O}(H) = (p^1_{HH} \circ \varphi)^*\mathcal{O}(H) \xrightarrow{\psi} (\varphi^* \circ (p^1_{HH})^*)\mathcal{O}(H),$$

and since we have that $(p^1_{HH})^*\mathcal{O}(H) = \mathcal{O}(H) \otimes \mathcal{O}(H)$, we get an isomorphism

$$\lambda : (p^1_{HH})^*\mathcal{O}(H) \xrightarrow{=} \mathcal{O}(H)\otimes\mathcal{O}(H) \xrightarrow{\varphi^*} \varphi^*(\mathcal{O}(H)\otimes\mathcal{O}(H)) \xrightarrow{\psi^{-1}} \mu_H^*\mathcal{O}(H).$$

It is now straightforward to check that $(\mathcal{O}(H), \lambda)$ is an $(H, \psi)-$equivariant coherent sheaf on H.

2) Since ι is affine, the commutative diagrams

$$\begin{tikzcd} H \times H \arrow[r, "p^1_{HH}"] \arrow[d, "\iota \times id_H"'] & H \arrow[d, "\iota"] \\ G \times H \arrow[r, "p^1_{GH}"'] & G \end{tikzcd} \qquad \begin{tikzcd} H \times H \arrow[r, "\mu_H"] \arrow[d, "\iota \times id_H"'] & H \arrow[d, "\iota"] \\ G \times H \arrow[r, "\mu_G"'] & G \end{tikzcd}$$

yield isomorphisms

$$(p^1_{GH})^*\iota_*\mathcal{O}(H) \xrightarrow{\cong} (\iota \times id_H)_*(p^1_{HH})^*\mathcal{O}(H) \tag{1}$$

and

$$(\iota \times id_H)_*\mu_H^*\mathcal{O}(H) \xrightarrow{\cong} \mu_G^*\iota_*\mathcal{O}(H) \tag{2}$$

("base change").

Let $\lambda : (p^1_{HH})^*\mathcal{O}(H) \xrightarrow{\cong} \mu_H^*\mathcal{O}(H)$ be the isomorphism constructed in Part 1. Since ι is $H-$equivariant, we get an isomorphism

$$(\iota \times id_H)_*(p^1_{HH})^*\mathcal{O}(H) \xrightarrow{(\iota \times id_H)_*(\lambda)} (\iota \times id_H)_*\mu_H^*\mathcal{O}(H). \tag{3}$$

It is now straightforward to check that the composition of isomorphisms (1), (3) and (2)

$$(p^1_{GH})^*\iota_*\mathcal{O}(H) \xrightarrow{\cong} \mu_G^*\iota_*\mathcal{O}(H)$$

endows $\iota_*\mathcal{O}(H)$ with a structure of an $(H, \psi)-$equivariant coherent sheaf on G. Clearly, $\iota_*\mathcal{O}(H)$ is simple.

3) Consider the right action $id \times \mu : G \times G \times H \to G \times G$ of H on $G \times G$. Since $M \in \mathrm{Coh}_f^{(H,\psi)}(G)$ it is clear that $X \boxtimes M \in \mathrm{Coh}(G \times G)$ is an $(H, \psi)-$equivariant coherent sheaf on $G \times G$. But since $m : G \times G \to G$ is $H-$equivariant, m_* carries $(H, \psi)-$equivariant coherent sheaves on $G \times G$ to $(H, \psi)-$equivariant coherent sheaves on G. $\square$

3.3. **Exact module categories.** Let G, H and ψ be as in 3.1-3.2, and consider the coalgebra $\mathcal{O}(H)_\psi$ in $\mathrm{Coh}(G)$. Let $\widehat{\mathcal{O}(H)_\psi}$ be its profinite completion with respect to the algebra structure of $\mathcal{O}(H)$ (see Example 2.4); it is a coalgebra object in both $\mathrm{Pro}(\mathrm{Coh}(G))$ and $\mathrm{Pro}(\mathrm{Coh}_f(G))$. Let $\mathrm{Comod}_{\mathrm{Pro}(\mathrm{Coh}_f(G))}(\widehat{\mathcal{O}(H)_\psi})$ be the abelian category of right comodules over $\widehat{\mathcal{O}(H)_\psi}$ in $\mathrm{Pro}(\mathrm{Coh}_f(G))$.

Proposition 3.7. *Let G be an affine group scheme over k, let H be a closed group subscheme of G and let ψ be a normalized $2-$cocycle on H. The following hold:*

1) Set $\mathcal{M} := Coh_f^{(H,\psi)}(G)$*. The bifunctor*

$$\otimes^{\mathcal{M}} : Coh_f(G) \boxtimes \mathcal{M} \to \mathcal{M},\ X \boxtimes M \mapsto m_*(X \boxtimes M),$$

defines on $\mathcal{M}$ *a structure of an indecomposable* $Coh_f(G)-$*module category.*

2) Set $\mathcal{V} := Comod_{Pro(Coh_f(G))}(\widehat{\mathcal{O}(H)_\psi})$*. The bifunctor*

$$\otimes^{\mathcal{V}} : Coh_f(G) \boxtimes \mathcal{V} \to \mathcal{V},\ X \boxtimes V \mapsto m_*(X \boxtimes V),$$

defines on $\mathcal{V}$ *a structure of an* $Coh_f(G)-$*module category.*

3) The categories $Coh_f^{(H,\psi)}(G)$ *and* $Comod_{Pro(Coh_f(G))}(\widehat{\mathcal{O}(H)_\psi})$ *are equivalent as module categories over* $Coh_f(G)$*. In particular,* $\overline{Hom}(\delta,\delta)$ *and* $\widehat{\mathcal{O}(H)_\psi}$ *are isomorphic as coalgebras in* $Pro(Coh_f(G))$*, where* $\delta = \delta_{(H,\psi)} := (\iota_H)_*\mathcal{O}(H) \in Coh_f^{(H,\psi)}(G)$*.*

Proof. 1) Since $m(m \times id) = m(id \times m)$ and ψ is a $2-$cocycle it follows from Lemma 3.6 that $\otimes^{\mathcal{M}}$ defines on $\mathcal{M}$ a structure of an $\mathrm{Coh}_f(G)-$module category. Moreover, consider the sheaf $\delta = \delta_{(H,\psi)} := (\iota_H)_*\mathcal{O}(H) \in \mathcal{M}$; it is a simple object (see Lemma 3.6). Then $\mathrm{Coh}_f(H)$ is the subcategory of $\mathrm{Coh}_f(G)$ consisting of those objects X for which $X \otimes^{\mathcal{M}} \delta$ is a multiple of δ, and any object $M \in \mathcal{M}$ is of the form $X \otimes^{\mathcal{M}} \delta$ for some $X \in \mathrm{Coh}_f(G)$. In particular, δ generates $\mathcal{M}$, so $\mathcal{M}$ is indecomposable.

2) By definition, an object in $\mathcal{V}$ is a pair (V, ρ_V) consisting of an object $V \in \mathrm{Pro}(\mathrm{Coh}_f(G))$ and a morphism $\rho_V : V \to V \otimes \widehat{\mathcal{O}(H)_\psi}$ in $\mathrm{Pro}(\mathrm{Coh}_f(G))$ satisfying the comodule axioms. Clearly, if $X \in \mathrm{Coh}_f(G)$ then $m_*(X \boxtimes V) \in \mathrm{Pro}(\mathrm{Coh}_f(G))$ and $\rho_{m_*(X\boxtimes V)} := id_X \otimes \rho_V$ is a morphism in $\mathrm{Pro}(\mathrm{Coh}_f(G))$ defining on $m_*(X \boxtimes V)$ a structure of a right comodule over $\widehat{\mathcal{O}(H)_\psi}$.

3) For any $S \in \mathrm{Pro}(\mathrm{Coh}_f(G))$ there is a natural isomorphism

$$\mathrm{Hom}_{G\times H}(\mu^*(S), (p^1_{GH})^*(S)) \cong \mathrm{Hom}_G(S, \mu_*(p^1_{GH})^*(S))$$

("adjunction"). Since $\mu_*(p^1_{GH})^*(S) \cong S \otimes \widehat{\mathcal{O}(H)}$, we can assign to any isomorphism $\lambda : \mu^*(S) \to (p^1_{GH})^*(S)$ a morphism $\rho_\lambda : S \to S \otimes \widehat{\mathcal{O}(H)}$. It is now straightforward to verify that (S, λ^{-1}) is an $(H,\psi)-$equivariant sheaf on G if and only if $\rho_\lambda : S \to S \otimes \widehat{\mathcal{O}(H)_\psi}$ is a comodule map. $\square$

We are now ready to state and prove the main result of this section.

Theorem 3.8. *Let* G *be an affine group scheme over* k*. There is a bijection between conjugacy classes of pairs* (H,ψ) *and equivalence*

classes of indecomposable exact module categories over $Coh_f(G)$*, assigning* (H,ψ) *to* $Coh_f^{(H,\psi)}(G)$*.*

Proof. Let $\mathrm{Coh}_f^{(H,\psi)}(G)$ be the indecomposable module category over $\mathrm{Coh}_f(G)$ constructed in Proposition 3.7. We have to show that it is exact, i.e., that any additive module functor $F:\mathrm{Coh}_f^{(H,\psi)}(G)\to\mathcal{M}$ from $\mathrm{Coh}_f^{(H,\psi)}(G)$ to any other $\mathcal{C}-$module category $\mathcal{M}$ is exact. Indeed, following the proof of [EO, Proposition 3.11], start with a short exact sequence

$$0\to X\to Y\to Z\to 0$$

in $\mathrm{Coh}_f^{(H,\psi)}(G)$, and assume on the contrary that the sequence

$$0\to F(X)\to F(Y)\to F(Z)\to 0$$

is not exact in $\mathcal{M}$. We may consider this sequence in the module category $\mathrm{Ind}(\mathcal{M})$ over $\mathrm{Ind}(\mathrm{Coh}_f(G))$. Since for any object $I\in\mathrm{Ind}(\mathrm{Coh}_f(G))$ the module functor $I\otimes^{\mathrm{Ind}(\mathcal{M})}?\ :\mathrm{Ind}(\mathcal{M})\to\mathrm{Ind}(\mathcal{M})$ is exact and $I\otimes^{\mathrm{Ind}(\mathcal{M})}X=0$ implies that $X=0$, we get that the sequence

$$0\to I\otimes F(X)\to I\otimes F(Y)\to I\otimes F(Z)\to 0$$

is not exact in $\mathrm{Ind}(\mathcal{M})$. Therefore, since F is a module functor, the sequence

$$0\to F(I\otimes X)\to F(I\otimes Y)\to F(I\otimes Z)\to 0$$

is not exact in $\mathrm{Ind}(\mathcal{M})$ too. But if I is injective, this sequence splits, since the sequence

$$0\to I\otimes X\to I\otimes Y\to I\otimes Z\to 0$$

splits in $\mathrm{Ind}(\mathrm{Coh}_f^{(H,\psi)}(G))$ by Proposition 2.6 (as F is additive). We thus got a contradiction, which proves that $\mathrm{Coh}_f^{(H,\psi)}(G)$ is exact.

Conversely, we have to show that any indecomposable exact module category $\mathcal{M}$ over $\mathrm{Coh}_f(G)$ is of the form $\mathrm{Coh}_f^{(H,\psi)}(G)$. Indeed, let $\delta\in\mathcal{M}$ be a simple object generating $\mathcal{M}$ (such δ exists by Corollary 2.9), and consider the full subcategory

$$\mathcal{C}:=\{X\in\mathrm{Coh}_f(G)\mid X\otimes^{\mathcal{M}}\delta=\dim_k(X)\delta\}$$

of $\mathrm{Coh}_f(G)$. Clearly, $\mathcal{C}$ is a tensor subcategory of $\mathrm{Coh}_f(G)$, so there exists a closed group subscheme H of G such that $\mathcal{C}\cong\mathrm{Coh}_f(H)$ as tensor categories. Moreover, the functor

$$F:\mathcal{C}\to\mathrm{Vec},\ F(X)=\mathrm{Hom}_{\mathcal{M}}(\delta,X\otimes^{\mathcal{M}}\delta),$$

together with the tensor structure $F(\cdot)\otimes F(\cdot)\xrightarrow{\cong}F(\cdot\otimes\cdot)$ coming from the associativity constraint, is a fiber functor. But, letting $\overline{X}$ denote the

underlying vector space of X (where we view X as an $\mathcal{O}(G)-$module), we see that $F(X) = \overline{X}$. We therefore get a functorial isomorphism $\overline{X} \otimes \overline{Y} \xrightarrow{\cong} \overline{X \otimes Y}$, which is nothing but an invertible element ψ of $\mathcal{O}(H) \otimes \mathcal{O}(H) = \mathcal{O}(H \otimes H)$ taking values in $\mathbb{G}_m(k)$. Clearly, ψ is a twist for $\mathcal{O}(H)$. Finally, since for any $X \in \mathrm{Coh}_f(H)$

$$\overline{X} = F(X) = \mathrm{Hom}_{\mathcal{M}}(\delta, X \otimes^{\mathcal{M}} \delta) = \mathrm{Hom}_{\mathrm{Pro}(\mathrm{Coh}_f(G))}(\overline{\mathrm{Hom}}(\delta, \delta), X),$$

Yoneda's lemma implies that the two coalgebras $\overline{\mathrm{Hom}}(\delta, \delta)$ and $\widehat{\mathcal{O}(H)_\psi}$ in $\mathrm{Pro}(\mathrm{Coh}_f(G))$ are isomorphic. But this implies that $\mathcal{M}$ is equivalent to $\mathrm{Comod}_{\mathrm{Pro}(\mathrm{Coh}_f(G))}(\widehat{\mathcal{O}(H)_\psi})$ as a $\mathrm{Coh}_f(G)-$module category (as $\mathcal{M}$ is indecomposable, exact and generated by δ), hence also to $\mathrm{Coh}_f^{(H,\psi)}(G)$ by Proposition 3.7. We are done. □

4. Exact module categories over Rep(G)

We keep the notation from Section 3. Set $\mathcal{M}(H, \psi) := \mathrm{Coh}_f^{(H,\psi)}(G)$, and recall that $\mathcal{M}(G, 1) = \mathrm{Vec}$.

4.1. **Module categories.** Recall that if $\mathcal{C}$ is a rigid tensor category and $\mathcal{M}$ is an exact module category over $\mathcal{C}$ then the category $\mathcal{C}^*_{\mathcal{M}} := \mathrm{End}_{\mathcal{C}}(\mathcal{M})$ of $\mathcal{C}-$endofunctors of $\mathcal{M}$ is called the *dual* of $\mathcal{C}$ with respect to $\mathcal{M}$. One has that $\mathcal{M}$ is an exact module category over $\mathcal{C}^*_{\mathcal{M}}$.

Example 4.1. We have that the category $\mathrm{Coh}_f(G)^*_{\mathcal{M}(H,\psi)}$ is the tensor category of $((H, \psi), (H, \psi))-$biequivariant sheaves on G, supported on finitely many left $H-$cosets (equivalently, right $H-$cosets), with tensor product given by convolution of sheaves. In particular, $\mathrm{Coh}_f(G)^*_{\mathcal{M}(G,1)}$ and $\mathrm{Rep}(G)$ are equivalent as tensor categories, and so are $\mathrm{Rep}(G)^*_{\mathcal{M}(G,1)}$ and $\mathrm{Coh}_f(G)$. □

One obtains the following result exactly as in the case of finite groups (see [Os]).

Proposition 4.2. *The following hold:*

1) If $\mathcal{M}$ is an exact module category over $Coh_f(G)$ then the category of module functors $Fun_{Coh_f(G)}(\mathcal{M}(G, 1), \mathcal{M})$ is an exact module category over $Rep(G)$ via the composition of functors. Conversely, if $\mathcal{N}$ is an exact module category over $Rep(G)$ then the category of module functors $Fun_{Rep(G)}(\mathcal{M}(G, 1), \mathcal{N})$ is an exact module category over $Coh_f(G)$.

2) The above two assignments, considered as assignments between exact module categories $\mathcal{M}$ over $Coh_f(G)$ and exact module categories $\mathcal{N}$

over Rep(G) for which Fun$_{Rep(G)}(\mathcal{M}(G,1),\mathcal{N})$ is nonzero, are inverses one to each other. $\square$

Let us say that an exact module category $\mathcal{N}$ over $\mathrm{Rep}(G)$ is *geometrical* if it satisfies $\mathrm{Fun}_{\mathrm{Rep}(G)}(\mathcal{M}(G,1),\mathcal{N}) \neq 0$. In other words, geometrical exact module categories over $\mathrm{Rep}(G)$ are those exact module categories which come from exact module categories over $\mathrm{Coh}_f(G)$.

Remark 4.3. 1) If G is *not* finite, $\mathrm{Rep}(G)$ may very well have module categories (even fiber functors) which are not geometrical. For example, let $G := \mathbb{G}_a^2$ over $\mathbb{C}$, let $J := \exp(x \otimes y)$, where x, y are a basis of the Lie algebra $\mathbb{C}^2$ (this makes sense on $G-$modules, since on them x, y are nilpotent so the Taylor series for exponential terminates), and let $\mathcal{N}$ be the semisimple $\mathrm{Rep}(G)-$module category of rank 1 corresponding to the twist J. Then the twisted algebra $\mathcal{O}(G)_J$ is the Weyl algebra generated by x, y with $yx - xy = 1$, so it does not have finite dimensional modules. Hence, $\mathrm{Fun}_{\mathrm{Rep}(G)}(\mathcal{M}(G,1),\mathcal{N}) = 0$.

Note that there is no $2-$cocycle ψ with values in $\mathbb{G}_m$ (there is one with values in $\mathbb{G}_a$, namely, $\psi((x_1,x_2),(y_1,y_2)) = x_1y_2 - x_2y_1$, but to make it take values in $\mathbb{G}_m$, one needs to take exponential, which is not algebraic).

2) The classification of fiber functors on $\mathrm{Rep}(G)$, where G is a unipotent algebraic group, is given in [EG3]. See also [EG2] for the construction of fiber functors on $\mathrm{Rep}(G)$ for other algebraic groups. However, the classification of fiber functors is not known for SL_n, $n \geq 4$ (it is known for $n \leq 3$ [Oh1], [Oh2]).

We shall refer to a finite dimensional comodule over $\mathcal{O}(H)_\psi$ as an $(H,\psi)-$*representation* of H, and denote the category of finite dimensional comodules over $\mathcal{O}(H)_\psi$ by $\mathrm{Corep}(\mathcal{O}(H)_\psi)$. Equivalently, the $2-$cocycle ψ determines a central extension H_ψ of H by $\mathbb{G}_m$, and an $(H,\psi)-$representation of H is a representation of the group scheme H_ψ on which $\mathbb{G}_m$ acts with weight 1 (i.e., via the identity character). Let us denote the category of representation of the group scheme H_ψ on which $\mathbb{G}_m$ acts with weight 1 by $\mathrm{Rep}_1(H_\psi)$.

Theorem 4.4. *The module category Fun$_{Coh_f(G)}(\mathcal{M}(G,1),\mathcal{M}(H,\psi))$ over Rep(G) corresponding to the pair (H,ψ) is equivalent to the category Rep(H,ψ) of finite dimensional $(H,\psi)-$representations of H (i.e., to Corep$(\mathcal{O}(H)_\psi) \cong$ Rep$_1(H_\psi)$).*

Proof. First observe that $\mathcal{F} := \mathrm{Fun}_{\mathrm{Coh}_f(G)}(\mathcal{M}(G,1),\mathcal{M}(H,\psi))$ and the category of $(G,(H,\psi))-$biequivariant sheaves on G are equivalent as abelian categories. Indeed, since $\mathcal{M}(G,1) = \mathrm{Vec}$, a functor in $\mathcal{F}$ is

just an $(H,\psi)-$equivariant sheaf X on G. The fact that the functor is a $\mathrm{Coh}_f(G)-$module functor gives X a commuting $G-$equivariant structure for the left action of G on itself, i.e., X is $(G,(H,\psi))-$biequivariant. Conversely, it is clear that any $(G,(H,\psi))-$biequivariant sheaf on G defines a functor in $\mathcal{F}$.

Now, if X is a $(G,(H,\psi))-$biequivariant sheaf on G then the inverse image sheaf $e^*(X)$ on $\mathrm{Spec}(k)$ ("the stalk at 1") acquires a structure of an $(H,\psi)-$representation via the action of the element (h,h^{-1}) in $G\times H$, i.e., it is an object in $\mathrm{Rep}(H,\psi)$. We have thus defined a functor $\mathcal{C}\to\mathrm{Rep}(H,\psi)$, $X\mapsto e^*(X)$.

Conversely, an $(H,\psi)-$representation V can be spread out over G and made into a $(G,(H,\psi))-$biequivariant sheaf X on G, with global sections $\mathcal{O}(G)\otimes_k V$. We have thus defined a functor $\mathrm{Rep}(H,\psi)\to\mathcal{C}$, $V\mapsto\mathcal{O}(G)\otimes_k V$.

Finally, it is straightforward to verify that the two functors constructed above are inverses of each other. □

4.2. **Semisimple module categories of rank** 1. Recall that the set of equivalence classes of semisimple module categories over $\mathrm{Rep}(G)$ of rank 1 is in bijection with the set of equivalence classes of tensor structures on the forgetful functor $\mathrm{Rep}(G)\to\mathrm{Vec}$. Therefore, Proposition 4.2 and Theorem 4.4 imply that the conjugacy class of any pair (H,ψ) for which the category $\mathrm{Corep}(\mathcal{O}(H)_\psi)$ is semisimple of rank 1 gives rise to an equivalence class of a tensor structure on the forgetful functor $\mathrm{Rep}(G)\to\mathrm{Vec}$. Clearly, for such pair (H,ψ), H must be a *finite* group subscheme of G (as a simple coalgebra must be finite dimensional). This observation suggests the following definition.

Definition 4.5. Let H be a finite group scheme over k. We call a $2-$cocycle $\psi:H\times H\to\mathbb{G}_m$ (equivalently, a twist ψ for $\mathcal{O}(H)=k[H]^*$) *nondegenerate* if the category $\mathrm{Corep}(\mathcal{O}(H)_\psi)$ of finite dimensional comodules over $\mathcal{O}(H)_\psi$ is equivalent to Vec (i.e., if the coalgebra $\mathcal{O}(H)_\psi$ is simple).

We thus have the following corollary.

Corollary 4.6. *The conjugacy class of a pair (H,ψ), where H is a finite closed group subscheme of G and $\psi:H\times H\to\mathbb{G}_m$ is a nondegenerate $2-$cocycle, gives rise to an equivalence class of a Hopf $2-$cocycle for $\mathcal{O}(G)$.* □

Remark 4.7. Finite group schemes having a nondegenerate $2-$cocycle may be called group schemes of *central type* in analogy with finite abstract groups.

5. GROUP SCHEME THEORETICAL CATEGORIES

The classes of rigid tensor categories $\mathrm{Rep}(G)$ and $\mathrm{Coh}_f(G)$ can be extended to a larger class of *group scheme theoretical categories*, exactly in the same way as it is done for finite groups [Os].

More precisely, let G be an affine group scheme over our field k and let $\omega \in H^3(G, \mathbb{G}_m)$ be a normalized $3-$cocycle. The proof of the following lemma is straightforward.

Lemma 5.1. *The category $Coh_f(G)$ (resp., $Coh(G)$) with tensor product given by convolution of sheaves and associativity constraint given by the action of ω (viewed as an invertible element in $\mathcal{O}(G)^{\otimes 3}$) is a rigid tensor category (resp., tensor category).* $\square$

Let us denote the rigid tensor category (resp., tensor category) from Lemma 5.1 by $\mathrm{Coh}_f(G, \omega)$ (resp., $\mathrm{Coh}(G, \omega)$).

Let H be a closed group subscheme of G and let $\psi \in C^2(H, \mathbb{G}_m)$ be a normalized $2-$cochain such that $d\psi = \omega_{|H}$. Let $\mathrm{Coh}_f^{(H,\psi)}(G, \omega)$ be the category of $(H, \psi)-$equivariant coherent sheaves on (G, ω); it is defined similarly to $\mathrm{Coh}_f^{(H,\psi)}(G)$ (the case $\omega = 1$) with the obvious adjustments. The proof of the following lemma is similar to the proof of Lemma 3.6.

Lemma 5.2. *The category $Coh_f^{(H,\psi)}(G, \omega)$ admits a structure of an indecomposable exact module category over $Coh_f(G, \omega)$ given by convolution of sheaves.* $\square$

The proof of the following classification result is similar to the proof of Theorem 3.8.

Theorem 5.3. *There is a bijection between conjugacy classes of pairs (H, ψ) and equivalence classes of indecomposable exact module categories over $Coh_f(G, \omega)$, assigning (H, ψ) to $Coh_f^{(H,\psi)}(G, \omega)$.* $\square$

Let us denote by $\mathcal{C}(G, H, \omega, \psi)$ the dual category of $\mathrm{Coh}_f(G, \omega)$ with respect to its indecomposable exact module category $\mathcal{M}(H, \psi)$. Then $\mathcal{C}(G, H, \omega, \psi)$ is the tensor category of $((H, \psi), (H, \psi))-$biequivariant sheaves on (G, ω), supported on finitely many left $H-$cosets (equivalently, right $H-$cosets), with tensor product given by convolution of sheaves.

Definition 5.4. A rigid tensor category which is tensor equivalent to some $\mathcal{C}(G, H, \omega, \psi)$ as above is called a group scheme theoretical category.

Example 5.5. Clearly, both $\mathrm{Rep}(G)$ and $\mathrm{Coh}_f(G,\omega)$ are group scheme theoretical categories. Another example is the center $\mathcal{Z}(\mathrm{Coh}_f(G))$ of $\mathrm{Coh}_f(G)$. Indeed, it is tensor equivalent to $\mathcal{C}(G\times G, G, 1, 1)$, where G is considered as a closed group subscheme of $G\times G$ via the diagonal morphism $\Delta : G\to G\times G$.

Let us give an instructive example of a center of $\mathrm{Coh}_f(G)$. Suppose that G is a semisimple adjoint algebraic group over $\mathbb{C}$ (i.e., with trivial center). Let $\mathfrak{g}^* = \mathrm{Lie}(G)^*$ be the coadjoint representation, regarded as a commutative algebraic group (multiple of $\mathbb{G}_a$). Then the center $\mathcal{Z}(\mathrm{Coh}_f(G))$ is braided equivalent to the category $\mathrm{Rep}(G\ltimes\mathfrak{g}^*)$ of finite dimensional rational representations of $G\ltimes\mathfrak{g}^*$ equipped with its natural (nonsymmetric) braided structure.

The following extends Proposition 4.2 (see also Remark 6.2 in the next section).

Proposition 5.6. *The following hold:*

1) If $\mathcal{M}$ is an exact module category over $Coh_f(G,\omega)$ then the category of module functors $Fun_{Coh_f(G,\omega)}(\mathcal{M}(H,\psi),\mathcal{M})$ is an exact module category over $\mathcal{C}(G,H,\omega,\psi)$ via the composition of functors. Conversely, if $\mathcal{N}$ is an exact module category over $\mathcal{C}(G,H,\omega,\psi)$ then the category of module functors $Fun_{\mathcal{C}(G,H,\omega,\psi)}(\mathcal{M}(H,\psi),\mathcal{N})$ is an exact module category over $Coh_f(G,\omega)$.

2) The above two assignments, considered as assignments between exact module categories $\mathcal{M}$ over $Coh_f(G,\omega)$ and exact module categories $\mathcal{N}$ over $\mathcal{C}(G,H,\omega,\psi)$ for which $Fun_{\mathcal{C}(G,H,\omega,\psi)}(\mathcal{M}(G,1),\mathcal{N})$ is nonzero, are inverses one to each other. □

6. Exact module categories over finite group schemes

In this section G will denote a *finite* group scheme over k.

6.1. **Module categories.** Thanks to [EO], Proposition 4.2 can be strengthened in the finite case to give a canonical bijection between exact module categories over $\mathrm{Coh}_f(G) = \mathrm{Coh}(G)$ and $\mathrm{Rep}(G)$ (i.e., for finite group schemes, any exact module category over $\mathrm{Rep}(G)$ is geometrical).

Theorem 6.1. *Let G be a finite group scheme over k. If $\mathcal{M}$ is an exact module category over $Coh(G)$ then the category of module functors $Fun_{Coh(G)}(\mathcal{M}(G,1),\mathcal{M})$ is an exact module category over $Rep(G)$ via the composition of functors. Conversely, if $\mathcal{M}$ is an exact module category over $Rep(G)$ then $Fun_{Rep(G)}(\mathcal{M}(G,1),\mathcal{M})$ is an exact module category over $Coh(G)$. Moreover, these two assignments are inverses*

one to each other. In particular, the equivalence classes of indecomposable exact module categories over Rep(G) *=* *Rep*$(k[G])$ *are parameterized by the conjugacy classes of pairs* (H,ψ)*, where* H *is a closed group subscheme of* G *and* $\psi : H\times H\to \mathbb{G}_m$ *is a* $2-$*cocycle.* □

Remark 6.2. More generally, the equivalence classes of indecomposable exact module categories over $\mathcal{C}(G,H,\omega,\psi)$ are parameterized by the conjugacy classes of pairs (H',ψ'), where H' is a group subscheme of G and $\psi'\in C^2(H',\mathbb{G}_m)$ satisfies $d\psi'=\omega_{|H'}$.

6.2. **Twists for** $k[G]$**.** By [AEGN], there is a bijection between nondegenerate twists for $k[G]$ and nondegenerate twists for $\mathcal{O}(G)$. Hence, as a consequence of Theorem 6.1, we deduce the following strengthening of Corollary 4.6.

Corollary 6.3. *Let* G *be a finite group scheme over* k*. The following four sets are in canonical bijection one with the other:*

1) The set of equivalence classes of tensor structures on the forgetful functor on Rep(G)*.*

2) The set of gauge equivalence classes of twists for $k[G]$*.*

3) The set of conjugacy classes of pairs (H,ψ)*, where* H *is a closed group subscheme of* G *and* $\psi : H\times H\to\mathbb{G}_m$ *is a nondegenerate* $2-$*cocycle.*

4) The set of conjugacy classes of pairs (H,J)*, where* H *is a closed group subscheme of* G *and* J *is a nondegenerate twist for* $k[H]$*.* □

Remark 6.4. Corollary 6.3 was proved for etale group schemes in [Mov], [EG] and [AEGN].

6.3. **Minimal twists for** $k[G]$**.** Recall that a twist J for $k[G]$ is called *minimal* if the triangular Hopf algebra $(k[G]^J, J_{21}^{-1}J)$ is minimal, i.e., if the left (right) tensorands of $J_{21}^{-1}J$ span $k[G]$ [R].

Using Deligne's theorem [De], it is shown in [EG, AEGN] that a twist for a finite abstract group is minimal if and only if it is nondegenerate. In this section we show that the same holds for any finite group scheme, without using Deligne's theorem. In order to achieve it, we shall need the following result about quotients of Tannakian categories, which is of interest by itself.

Proposition 6.5. *Let* G *be a finite group scheme over* k*, let* $\mathcal{C}$ *be a symmetric rigid tensor category over* k*, and suppose there exists a surjective symmetric tensor functor* $F : Rep(G)\to\mathcal{C}$*. Then there exists a closed group subscheme* H *of* G *such that* $\mathcal{C}$ *is equivalent to Rep*(H) *as a symmetric rigid tensor category and* $Forget_G \cong Forget_H\circ F$*.*

Proof. Consider the image $F(\mathcal{O}(G))$ of the commutative unital algebra object $\mathcal{O}(G)$ in $\mathrm{Rep}(G)$; it is a commutative unital algebra object in $\mathcal{C}$. Let $I \in \mathcal{C}$ be a maximal ideal subobject of $F(\mathcal{O}(G))$, and set $R := F(\mathcal{O}(G))/I$. Then R is a commutative unital algebra object in $\mathcal{C}$. Let $\mathrm{Mod}_{\mathcal{C}}(R)$ be the category of modules in $\mathcal{C}$ over R. Clearly, R is a simple object in $\mathrm{Mod}_{\mathcal{C}}(R)$, so $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, R) = \mathrm{Hom}_R(R, R) = k$.

Observe that for any $X \in \mathcal{C}$, there exists a finite dimensional vector space $\overline{X}$ such that $X \otimes R \cong \overline{X} \otimes_k R$ as modules over R (i.e., $X \otimes R$ is free). Indeed, this follows since $\mathcal{O}(G) \in \mathrm{Rep}(G)$ has this property and F is surjective. Therefore, since $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, R) = k$, it follows that $\mathrm{Hom}_R(R, X \otimes R) = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, X \otimes R)$ canonically by Frobenius reciprocity, which implies that there is a canonical isomorphism $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, X \otimes R) \otimes_k R \xrightarrow{\cong} X \otimes R$. Hence the functor

$$L : \mathcal{C} \to \mathrm{Vec},\ X \mapsto \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, X \otimes R),$$

together with the tensor structure given by

$$\begin{aligned} L(X \otimes Y) &= \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, (X \otimes Y) \otimes R) \\ &\xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, X \otimes (L(Y) \otimes_k R)) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, (L(X) \otimes_k L(Y)) \otimes_k R)) \\ &\xrightarrow{\cong} L(X) \otimes_k L(Y), \end{aligned}$$

is a fiber (= exact tensor) functor on $\mathcal{C}$. But then a standard argument (see e.g., [DM]) yields that $\mathcal{C}$ is equivalent to $\mathrm{Rep}(A)$ for some finite dimensional Hopf algebra A over k, as a rigid tensor category. Hence, there exists an injective homomorphism $A \xrightarrow{1-1} k[G]$ of Hopf algebras, and the result follows. $\square$

Remark 6.6. Proposition 6.5 holds for any affine group scheme over k (i.e., not necessarily finite). Namely, quotients of Tannakian categories are Tannakian. The proof is essentially the same, except that $\mathcal{O}(G)$ and its image under (the extension of) F are Ind objects, so certain adaptations are required.

We can now state and prove the main result of this section.

Proposition 6.7. *Let G be a finite group scheme over k, and let J be a twist for $k[G]$. Then J is minimal if and only if it is nondegenerate.*

Proof. Suppose J is minimal. By Corollary 6.3, there exist a closed group subscheme $\overline{H}$ of H and a nondegenerate twist $\overline{J}$ for $k[\overline{H}]$ such that the image of $\overline{J}$ under the embedding $k[\overline{H}]_{\overline{J}} \hookrightarrow k[H]_J$ is J. Since J is minimal and $\overline{H} \subseteq H$, it follows that $\overline{H} = H$.

Conversely, suppose J is nondegenerate. Let $(A, J_{21}^{-1}J)$ be the minimal triangular Hopf subalgebra of $(k[G]^J, J_{21}^{-1}J)$. The restriction functor $\mathrm{Rep}(G) \to \mathrm{Rep}(A)$ is a surjective symmetric tensor functor. Thus

by Proposition 6.5, Rep(A) is equivalent to Rep(H), as a symmetric tensor category, for some closed group subscheme H of G. Now, it is a standard fact (see e.g., [Ge]) that such an equivalence functor gives rise to a twist $I \in k[H]^{\otimes 2}$ and an isomorphism of triangular Hopf algebras $(k[H]^I, I_{21}^{-1}I) \xrightarrow{\cong} (A, J_{21}^{-1}J)$.

We therefore get an injective homomorphism of triangular Hopf algebras $(k[H]^I, I_{21}^{-1}I) \xrightarrow{1-1} (k[G]^J, J_{21}^{-1}J)$, which implies that JI^{-1} is a symmetric twist for $k[G]$. But by [DM, Theorem 3.2], this implies that JI^{-1} is gauge equivalent to $1 \otimes 1$. Therefore, the triangular Hopf algebras $(k[G]^{JI^{-1}}, I_{21}J_{21}^{-1}JI^{-1})$ and $(k[G], 1 \otimes 1)$ are isomorphic. In other words, $(k[G]^I, I_{21}^{-1}I)$ and $(k[G]^J, J_{21}^{-1}J)$ are isomorphic as triangular Hopf algebras, i.e., the pairs (G, J) and (H, I) are conjugate. We thus conclude from Corollary 6.3 that $H = G$, and hence that J is a minimal twist, as required. □

6.4. **The commutative case.** Let A be a finite commutative group scheme over k and let A^D be its Cartier dual (see Section 2.1). By definition, $k[A] = \mathcal{O}(A^D)$ and $k[A^D] = \mathcal{O}(A)$. Therefore, Corollary 6.3 implies the following.

Proposition 6.8. *There is a canonical isomorphism of abelian groups between the group of gauge equivalence classes of twists for $k[A]$ and the group $H^2(A^D, \mathbb{G}_m)$.* □

Corollary 6.9. *Suppose that either $A = A_{ec}$ or $A = A_{ce}$. Then the equivalence classes of indecomposable exact module categories over Rep(A) are in bijection with the conjugacy classes of closed group subschemes of A. In particular, the trivial twist is the only twist for $k[A]$, i.e., the forgetful functor on Rep(A) has only the trivial tensor structure.*

Proof. By Proposition 6.8, it is sufficient to show that in both cases $H^2(A, \mathbb{G}_m) = 0$. Indeed, consider the group homomorphism

$$H^2(A, \mathbb{G}_m) \to \mathrm{Hom}(A \times A, \mathbb{G}_m),\ \psi \mapsto \psi_{21}^{-1}\psi;$$

it is well defined since for any two choices ψ_1, ψ_2 the $2-$cocycle $\psi_1\psi_2^{-1}$ is symmetric, and it is known that $H_s^2(A, \mathbb{G}_m) = \mathrm{Ext}^1(A, \mathbb{G}_m) = 0$ (see, e.g., [Mum]). Clearly, its image is contained in the group of skew-symmetric bilinear forms on A, i.e., is contained in $\mathrm{Hom}(A, A^D)$. But since $\mathrm{Hom}(A_{ec}, A_{ce}) = 0$, $\mathrm{Hom}(A, A^D) = 0$, so the above homomorphism is trivial. This means that $H^2(A, \mathbb{G}_m) = H_s^2(A, \mathbb{G}_m) = 0$, as claimed. □

In contrast, the cases $A = A_{ee}$ (see Remark 6.4) and $A = A_{cc}$ (see Example 6.13) are more interesting, as demonstrated also by the following proposition.

Proposition 6.10. *Let $\psi \in H^2(A \times A^D, \mathbb{G}_m)$ be the class represented by the $2-$cocycle given by $\psi((a_1, f_1), (a_2, f_2)) =< f_1, a_2 >$, where $<,>$: $A^D \times A \to \mathbb{G}_m$ is the canonical pairing. Then ψ is nondegenerate, i.e., it corresponds to a nondegenerate twist for $k[A \times A^D]$.*

Proof. It is straightforward to verify that the corresponding twist for $\mathcal{O}(A \times A^D)$ (which we shall also denote by ψ) is given by $\psi = \sum f_i \otimes a_i$, where f_i and a_i are dual bases of $\mathcal{O}(A)$ and $\mathcal{O}(A)^*$, respectively. But $(\mathcal{O}(A \times A^D)_\psi)^*$ is an Heisenberg double, hence a simple algebra by [Mon, Corollary 9.4.3] (see [AEGN]), so ψ is nondegenerate. $\square$

6.5. $p-$**Lie algebras.** Assume that k has characteristic $p > 0$. In the case of $p-$Lie algebras (see Section 2.2), Theorem 6.1 and Corollary 6.3 translate into the following result.

Theorem 6.11. *Let $\mathfrak{g}$ be a finite dimensional $p-$Lie algebra over k. The equivalence classes of indecomposable exact module categories over Rep$(\mathfrak{g})$ are parameterized by the conjugacy classes of pairs $(\mathfrak{h}, \psi)$, where $\mathfrak{h}$ is a $p-$Lie subalgebra of $\mathfrak{g}$ and ψ is a Hopf $2-$cocycle for $u(\mathfrak{h})$. In particular, the gauge equivalence classes of twists for $u(\mathfrak{g})$ are in bijection with conjugacy classes of pairs $(\mathfrak{h}, J)$, where $\mathfrak{h}$ is a $p-$Lie subalgebra of $\mathfrak{g}$ and J is a nondegenerate twist for $u(\mathfrak{h})$.* $\square$

Example 6.12. (Semisimple $p-$Lie algebras) Let $\mathfrak{t}$ be a torus, i.e., the $p-$Lie algebra of a connected diagonalizable group scheme. Then Corollary 6.9 tells us that the forgetful functor on Rep$(\mathfrak{t})$ has only the trivial tensor structure, which is in contrast with the etale case.

Example 6.13. Let $\mathfrak{a}$ be the $2-$dimensional abelian $p-$Lie algebra with basis h, x such that $h^p = 0$ and $x^p = 0$ (it is the $p-$Lie algebra of the group scheme $\alpha_p \times \alpha_p$). Then it is straightforward to verify that

$$J := \exp(h \otimes x) = \sum_{i=0}^{p-1} \frac{h^i \otimes x^i}{i!}$$

is a nondegenerate twist for $u(\mathfrak{a})$. In fact, the algebra $(u(\mathfrak{a})_J)^*$ is isomorphic to the truncated Weyl algebra $k[x, y]/(xy - yx - 1, x^p, y^p)$, which is known to be a simple algebra ([S, p.73]).

Example 6.14. Let $\mathfrak{g}$ be the unique $2-$dimensional nonabelian $p-$Lie algebra with basis x, y such that $[x, y] = y$, $x^p = x$ and $y^p = 0$ (it is the $p-$Lie algebra of the Frobenius kernel of the group scheme $\mathbb{G}_m \ltimes \mathbb{G}_a$

of automorphisms of the affine line $\mathbb{A}^1$). It is straightforward to verify that the element

$$J := \sum_{i=0}^{p-1} \frac{x(x-1)\cdots(x-i+1)\otimes y^i}{i!}$$

is a nondegenerate twist for $u(\mathfrak{g})$, and that $(u(\mathfrak{g})^J, J_{21}^{-1}J)$ is a noncommutative and noncocommutative minimal triangular Hopf algebra of dimension p^2.

Example 6.15. (Frobenius $p-$Lie algebras) Let $\mathfrak{g}$ be a finite dimensional *Frobenius* $p-$Lie algebra over k. By definition, this means that there exists a linear functional $\xi \in \mathfrak{g}^*$ such that the bilinear form $(x, y) \mapsto \xi([x, y])$ on $\mathfrak{g}$ is nondegenerate. By a well known result of Premet-Skryabin [PS], the associated reduced universal enveloping algebra $u_\xi(\mathfrak{g})$ of $\mathfrak{g}$ is a simple algebra. Therefore any Frobenius $p-$Lie algebra possesses a nondegenerate twist.

Since there are nonsolvable Frobenius $p-$Lie algebras, we see that there exist finite group schemes of central type which are *not* solvable (unlike in the etale case [EG]). For example, the $6-$dimensional $p-$Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}_3$, consisting of the matrices with zero last row, is not solvable but is Frobenius (e.g., let $\xi \in \mathfrak{g}^*$ be defined on the standard basis E_{ij} by $\xi(E_{12}) = \xi(E_{23}) = 1$ and $\xi(E_{11}) = \xi(E_{13}) = \xi(E_{21}) = \xi(E_{22}) = 0$).

Example 6.16. (The Witt $p-$Lie algebra) Let $\mathfrak{w}$ be the $p-$dimensional $p-$Lie algebra with basis x_i, $i \in \mathbb{F}_p$, such that $[x_i, x_j] = (j-i)x_{i+j}$, $x_0^p = x_0$ and $x_i^p = 0$ for $i \neq 0$. Note that for any $i \neq 0$, the elements $x := i^{-1}x_0$, $y := ix_i$ span a $2-$dimensional nonabelian $p-$Lie subalgebra of $\mathfrak{w}$. We thus obtain twists $J(i)$ for $u(\mathfrak{w})$, $i \in \mathbb{F}_p^\times$, as in Example 6.14, and hence p^p-dimensional noncommutative and noncocommutative triangular Hopf algebras $u(\mathfrak{w})^{J(i)}$. (See [Gr] for similar results.)

7. ISOCATEGORICAL FINITE GROUP SCHEMES

Following [EG1], we say that two finite group schemes G_1, G_2 over k are *isocategorical* if $\text{Rep}(G_1)$ is equivalent to $\text{Rep}(G_2)$ as a tensor category (without regard for the symmetric structure). Then G_1, G_2 are isocategorical if and only if the Hopf algebras $k[G_1]$, $k[G_2]$ are twist equivalent [EG1]. The construction of all finite group schemes isocategorical to a fixed finite group scheme is essentially identical to the construction given in [EG1] for etale groups. Therefore, for the sake of completeness only, we shall describe the construction and sketch the

proof given in [EG1] (see also [Da1, Da2]) that this construction is exhaustive.

7.1. **The construction of isocategorical finite group schemes.** Let G be finite group scheme over k, A a commutative normal closed group subscheme of G, and set $K := G/A$. Let $R : A^D \times A^D \to \mathbb{G}_m$ be a $G-$equivariant nondegenerate skew-symmetric (i.e., $R(a,a) = 0$ for all $a \in A^D$) bilinear form on A^D. It is known that the image of the group homomorphism

$$H^2(A^D, \mathbb{G}_m) \to \mathrm{Hom}(A^D \times A^D, \mathbb{G}_m),\ \psi \mapsto \psi\psi_{21}^{-1},$$

is the group of skew-symmetric bilinear forms on A^D. Therefore, the form R defines a class in $H^2(A^D, \mathbb{G}_m)^K$ represented by any $2-$cocycle $J \in Z^2(A^D, \mathbb{G}_m)$ such that $R = JJ_{21}^{-1}$.

Let

$$\tau : H^2(A^D, \mathbb{G}_m)^K \to H^2(K, A)$$

be the homomorphism defined as follows. For $c \in H^2(A^D, \mathbb{G}_m)^K$, let J be a $2-$cocycle representing c. Then for any $g \in K$, the $2-$cocycle $J^g J^{-1}$ is a coboundary. Choose a cochain $z(g) : A^D \to \mathbb{G}_m$ such that $dz(g) = J^g J^{-1}$, and let

$$\tilde{b}(g,h) := z(gh)z(g)^{-1}(z(h)^g)^{-1}.$$

Then for any $g, h \in K$, the function $\tilde{b}(g,h) : A^D \to \mathbb{G}_m$ is a group homomorphism, i.e., $\tilde{b}(g,h)$ belongs to the group A. Thus, $\tilde{b}$ can be regarded as a $2-$cocycle of K with coefficients in A. So $\tilde{b}$ represents a class b in $H^2(K, A)$, which depends only on c and not on the choices we made. So we define τ by $\tau(c) = b$.

Now, let $b := \tau(R)$, and let $\tilde{b}$ be any cocycle representing b. For any $\gamma \in G$, let $\bar{\gamma}$ be the image of γ in K. Introduce a new multiplication law $*$ on the scheme G by

$$\gamma_1 * \gamma_2 := \tilde{b}(\bar{\gamma}_1, \bar{\gamma}_2)\gamma_1\gamma_2.$$

It is easy to show that this multiplication law introduces a new group scheme structure on G, which (up to an isomorphism) depends only on b and not on $\tilde{b}$. Let us call this finite group scheme G_b.

Theorem 7.1. *The following hold:*

1) The finite group scheme G_b is isocategorical to G.

2) Any finite group scheme isocategorical to G is obtained in this way.

In particular, two isocategorical finite group schemes are necessarily isomorphic as schemes (but not as groups [EG1]*).*

7.2. **Sketch of the proof of Theorem 7.1.** Suppose that G_1 and G_2 are isocategorical, and fix a twist J for $k[G_1]$ such that $k[G_1]^J$ and $k[G_2]$ are isomorphic as Hopf algebras (but *not* necessarily as triangular Hopf algebras). Clearly, the Hopf algebra $k[G_1]^J$ is cocommutative. Set, $R^J := J_{21}^{-1}J$.

Let $(k[G_1]^J)_{min} \subseteq k[G_1]^J$ be the minimal triangular Hopf subalgebra of the triangular Hopf algebra $(k[G_1]^J, R^J)$ [R]. Since $(k[G_1]^J)_{min}$ is isomorphic to its dual with opposite coproduct (via R^J), $(k[G_1]^J)_{min}$ is cocommutative and commutative. This implies that $(k[G_1]^J)_{min}$ is isomorphic to the group algebra $k[A]$ of a commutative group scheme A. Therefore, there exists a twist $J' \in (k[G_1]^J)_{min} \otimes (k[G_1]^J)_{min}$ such that $R^J = R^{J'}$. But this implies (exactly as in the proof of Proposition 3.4 in [EG1]) that there exists a twist $\widehat{J}$ for $k[G_1]$ such that $k[G_1]^J$ is isomorphic to $k[G_1]^{\widehat{J}}$ as triangular Hopf algebras, and $\widehat{J} \in (k[G_1]^{\widehat{J}})_{min} \otimes (k[G_1]^{\widehat{J}})_{min}$.

Thus, we can assume, without loss of generality, that $J \in (k[G_1]^J)_{min}^{\otimes 2}$. This implies that $(k[G_1]^J)_{min} = k[A]$, where A is a commutative closed group subscheme of G_1, and $J \in k[A] \otimes k[A]$.

Proposition 7.2. *The closed group subscheme A is normal in G_1 (i.e., $k[A]$ is invariant under the adjoint action Ad of $k[G_1]$ on itself), and the action of the group scheme $K := G_1/A$ on A by conjugation preserves R^J.*

Proof. By cocommutativity of $k[G_1]^J$, $J^{-1}\Delta(g)J = J_{21}^{-1}\Delta(g)J_{21}$ for all $g \in k[G_1]$, hence $\Delta(g)R^J = R^J\Delta(g)$ (here we use that $k[A]$ is commutative, so $R^J = JJ_{21}^{-1}$). But then, using the cocommutativity of $k[G_1]^J$ again, we get that R^J is invariant under the adjoint action of $k[G_1]$, i.e., $\mathrm{Ad}(g)R^J = \varepsilon(g)R^J$ for all $g \in k[G_1]$. Since the left (and right) tensorands of R^J span $k[A]$, the result follows. □

We can thus view J not only as a twist for $k[A]$ but also as a $2-$cocycle of A^D with values in $\mathbb{G}_m$, according to Proposition 6.8. For $g \in K$ let us write J^g for the action of g on J. Since R^J is invariant under G_1, $J^gJ^{-1} = J_{21}^gJ_{21}^{-1}$, which implies that the $2-$cocycle $J^gJ^{-1} : A^D \times A^D \to \mathbb{G}_m$ is symmetric. Hence there exists a cochain $z(g) : A^D \to \mathbb{G}_m$ (i.e., an invertible element in $\mathcal{O}(A^D) = k[A]$) such that $J^gJ^{-1} = dz(g)$.

Identifying $k[G_1]^J$ with $k[G_2]$, we can consider the morphism of schemes $\varphi : G_1 \to G_2$, $\varphi(\gamma) = z(\gamma)^{-1}\gamma$ (where by $z(\gamma)$ we mean $z(\gamma A)$). Then φ is bijective (with inverse $\varphi^{-1}(\gamma) = z(\gamma)\gamma$).

Finally, it is obvious from the definition of φ that

$$\varphi(\gamma_1)\varphi(\gamma_2) = \tilde{b}(\bar{\gamma}_1, \bar{\gamma}_2)\varphi(\gamma_1\gamma_2),$$

where $\tilde{b}(g,h) = z(gh)/z(g)z(h)^g \in k[A]^\times$. Furthermore, the morphism $\tilde{b} := \tilde{b}(g,h) : A^D \to \mathbb{G}_m$ is a group homomorphism, i.e., $\tilde{b} \in A$, and it is clear that $\tilde{b}$ is a $2-$cocycle of K with coefficients in A. Let b be the cohomology class of $\tilde{b}$ in $H^2(K,A)$. We have shown that

$$\varphi(\gamma_1 * \gamma_2) = \varphi(\gamma_1)\varphi(\gamma_2),$$

i.e., that φ is an isomorphism of group schemes $(G_1)_b \to G_2$. This completes the proof of Part 2 of Theorem 7.1, since by the definition of b we have $b = \tau(\overline{R^J})$.

Finally, Part 1 is essentially obvious from the above. Namely, if G is a finite group scheme, A its commutative normal closed group subscheme, $K := G/A$ and $b := \tau(\bar{R}) \in H^2(K,A)$, then choose a twist $J \in k[A]^{\otimes 2}$ such that $R = J_{21}^{-1}J$ and get that $k[G]^J$ is isomorphic as a Hopf algebra to $k[G_b]$ (so the group schemes G and G_b are isocategorical). $\square$

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Department of Mathematics, Technion-Israel Institute of Technology, Haifa 32000, Israel

E-mail address: gelaki@math.technion.ac.il